# Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces

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## Abstract

This paper is the third of a series on Hamiltonian stationary Lagrangian surfaces. We present here the most general theory, valid for any Hermitian symmetric target space. Using well-chosen moving frame formalism, we show that the equations are equivalent to an integrable system, generalizing the $\mathbb{C}^2$ subcase analyzed in [HR1]. It shares many features with the harmonic map equation of surfaces into symmetric spaces, allowing us to develop a theory close to Dorfmeister, Pedit and Wu's, including for instance a Weierstrass-type representation. Notice that this article encompasses the article mentioned above, although much fewer details will be given on that particular flat case.

## Introduction

Hamiltonian stationary Lagrangian submanifolds are Lagrangian submanifolds of a given symplectic manifold $M$ which are critical points of the volume functional with respect to a special class of infinitesimal variations preserving the Lagrangian constraint, namely the class of compactly supported Hamiltonian vector fields. This makes sense if $M$ is not only a symplectic but is also a Riemannian manifold. This is true in particular if $M$ is a Kähler manifold. The Euler-Lagrange equation has a particularly elegant formulation in terms of the so-called *Lagrangian angle* $\beta$, a $\mathbb{R}/2\pi\mathbb{Z}$-valued function defined along any Lagrangian submanifold. We may think of $\beta$ as part of the Gauss map. A Lagrangian surface is Hamiltonian stationary if and only if $\beta$ is harmonic.

This problem has been addressed recently by some authors like J. Wolfson [Wo]. It offers a nice generalization of the minimal surface problem which

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maybe shares more similarity with the constant mean curvature surfaces problem, since the Lagrangian constraint could be thought as a replacement for the volume constraint. Along these lines, Y. G. Oh studied as a particular solution the Clifford tori in $\mathbb{C}^2$ or $\mathbb{CP}^2$ and conjectured that these tori actually minimize the area among all tori which are isotopic by Hamiltonian deformation, a statement which looks like an isoperimetric inequality [O]. Other examples of Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ were found by I. Castro and F. Urbano [CaUr]. Another motivation for looking at this problem lies in its similarity to some models in incompressible elasticity [Wo, HR1]. Furthermore, the Hamiltonian stationary class includes as a subcase the class of special Lagrangian manifolds, which are calibrated manifolds introduced by R. Harvey and H. B. Lawson [HaL]. These are simply Lagrangian surfaces in a Calabi-Aubin-Yau 4-manifold with *constant* Lagrangian angle function $\beta$. In [ScWo], R. Schoen and J. Wolfson propose to produce special Lagrangian manifolds by first constructing a Hamiltonian stationary Lagrangian submanifold using analytical methods and then proving that, under some hypotheses, these submanifolds are in fact special Lagrangian. These questions are also strongly motivated by string theory and branes theory, where special Lagrangian submanifolds play an important role.

In [HR1] we considered Hamiltonian stationary Lagrangian surfaces in $\mathbb{R}^4 \simeq \mathbb{C}^2$ and showed that this problem is completely integrable, like the famous KdV equation or, in differential geometry, the equation for harmonic maps from a surface to a symmetric space. Our formalism was similar to the harmonic map theory, except that the connection form is not of the form $\alpha_\lambda := \lambda^{-1}\alpha'_1 + \alpha_0 + \lambda\alpha''_1$ but $\alpha_\lambda := \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2$, where $\lambda$ is a complex (spectral) parameter. The equations being slightly more linear than in the harmonic maps context, we were able to simplify the integrable system theory and propose a Weierstrass type representation formula much simpler than the one constructed by J. Dorfmeister, F. Pedit and H. Y. Wu in [DPW]. In [HR2] we further simplified these formulas using quaternions and compared them with a similar formula due to B. G. Konopelchenko, suggesting a hidden structure of spinors. Using these formulas, H. Anciaux recently showed estimate towards Oh's conjecture [A].

In the following paper, we shall show that the integrable system structure – with a family of connections of the type $\alpha_\lambda := \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2$ – persists if one replaces the ambient space by any two-dimensional Hermitian symmetric space. It should be noted that the Hermitian symmetric spaces are automatically Kähler-Einstein. They are of five types, namely : $\mathbb{R}^4 \simeq \mathbb{C}^2$, $\mathbb{CP}^2$ and its dual the complex hyperbolic space, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and its dual. Moreover they share a crucial algebraic property: the existence of an order four automorphism $\tau$, squaring to the usual symmetry $\sigma$, where $M$ is the quotient of all isometries over the fixed point set of $\sigma$. The specific properties of $\tau$ will be described below. As an application we show that the theory in [DPW] can be generalized (fully for compact spaces, partially with local versions for non compact spaces as in [H]) and that conformal parametrizations of such surfaces are constructed using holomorphic data. There is no doubt that it is

also possible to build a theory of finite type solutions (in particular for tori) using our equations, as was done in details in [HR1]. However, unlike in the harmonic map setting, the choice of appropriate moving frames plays a key role in our theory. We call them *Lagrangian framings*. We show how they help to characterize the Lagrangian angle and describe their properties in great details in section 1.2.

By the same token, we can extend our description to Hamiltonian stationary Lagrangian cones in $\mathbb{C}^3$, since their links (i.e. intersection with the sphere $S^5$) are stationary Legendrian surfaces which project down to Hamiltonian stationary Lagrangian surfaces in $\mathbb{CP}^2$ by the Hopf fibration. Note that special Lagrangian cones in $\mathbb{C}^3$ with toric links have been constructed by M. Haskins in [Has].

A last comment: as for harmonic maps, constant mean curvature surfaces in $\mathbb{R}^3$ or Willmore surfaces [H], the emergence of such a miraculous theory is linked to the existence of conjugate families of solutions (which here are obviously obtained by rotating the spectral parameter $\lambda$ in $S^1$). There is however a novelty with Hamiltonian stationary Lagrangian surfaces, residing in its different connection form $\alpha_\lambda$ involved. We may furthermore observe that this connection mixes spinor-like quantities ($\alpha_{-1}$, $\alpha_1$) and non spinor-like ones ($\alpha_{-2}$, $\alpha_0$), so we may ask whether there is some supersymmetric interpretation of that.

# 1   Moving frames and Lagrangian lifts

## 1.1   Lagrangian framings of Lagrangian surfaces in a Kähler 4-manifold

Let $M$ be a Kähler 4-manifold with almost complex structure $J$, and $B$ the principal $U(2)$-bundle of unitary frames on $M$ (with the obvious $U(2)$ action); define $B' = B/SU(2)$ as the quotient bundle (indeed a principal $U(1)$-bundle). Notice that $B'$ is diffeomorphic to (dual of) the canonical bundle on $M$. Let $f : L \to M$ be an immersion of a surface into $M$. The surface is Lagrangian if $f^*\omega = 0$ where $\omega$ is the Kähler form; that amounts to saying that for any $p \in M$, the tangent plane $T_pL$ at $p$ to $L$ is mapped by $J$ to its (Riemannian) orthogonal complement or, in other words, that any Riemannian-orthonormal basis $(e_1, e_2)$ of $T_pL$ is actually Hermitian-orthonormal in $TM$.

Hence for any $p \in L$ the set of all Riemannian orthonormal bases of the tangent plane $T_pL$ can be described as being $SO(2)_*(e_1, e_2)$ – i.e. some orbit of the action of $SO(2)$ – and this set is a subset of the set of all Hermitian-orthogonal bases at $p$, i.e. the orbit $U(2)_*(e_1, e_2)$. In between lies the orbit $SU(2)_*(e_1, e_2)$ which precisely interests us.

**Definition 1.1** *A (local) framing of the immersion $f$ is a (local) section of the unitary frame bundle $f^*B$. A framing $F$ of $f$ is called* Lagrangian[1] *if for any*

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[1]this choice ought to be compared to Darboux framings in the Riemannian setting, for a submanifold $N^n$ of $M^m$: a Darboux framing is an orthonormal framing such that the $n$

$p \in L$, $F(p)$ *is equivalent mod $SU(2)$ to an orthonormal basis $(e_1(p), e_2(p))$ of* $T_pL$.

Note that there may not exist global sections (Lagrangian or not) of $f^*B$, except in coordinate charts. However it is clear that all local Lagrangian framings lift a single section of the bundle $B'$ by the projection map $B \to B'$ and because of this uniqueness we obtain a globally defined section of $B'$.

**Definition 1.2** *The image of Lagrangian framings by the quotient map $B \to B'$ is called the* tangent section *of the bundle $B'$.*

If $M$ is Kähler-Einstein, we may describe Lagrangian framings more precisely. We follow here the exposition in [Wo] further refined to symmetric spaces. Let $K$ denote as is customary the canonical bundle of $M$. Note that $K$ is canonically isomorphic to $B'$. On the pull-back bundle $f^*K$ with associate pull-back metric, the first Chern form satisfies

$$f^*c_1 = f^*\text{Ric} = Rf^*\omega = 0.$$

(Here Ric denotes the Ricci form and $R$ the scalar curvature.) Hence $f^*K$ is flat and we can construct (local) parallel sections. Any parallel unit section $s_0$ defines a trivialization of $f^*K \simeq f^*B'$, in which the tangent section is described by a unit complex number $e^{i\beta}$. Another way to express that is by evaluating the parallel section $s_0$ of $f^*K$ against any orthonormal framing $e_1, e_2$ of $TL$

$$e^{i\beta} = s_0(e_1, e_2).$$

The real valued function $\beta$ is called the Lagrangian angle[2]. A different choice of $s_0$ will only change $\beta$ by a constant. A Lagrangian framing is a lift with the same Lagrangian angle as the tangent section. Finally we define the Maslov form as $\Theta = \frac{1}{\pi}d\beta$, and it is automatically closed. It is useful to have another expression of the Maslov form, namely $\Theta = \frac{1}{\pi}\iota_H\omega$ where $H$ is the mean curvature vector field. Hence Lagrangian surfaces with constant Lagrangian angle are automatically minimal and vice-versa. If furthermore the canonical bundle on $M$ is flat (as in $\mathbb{C}^2$ or more generally as in any Calabi-Aubin-Yau manifold), there is a globally defined Lagrangian angle obtained by pulling back a global parallel unit section of $K$ (e.g. $dz^1 \wedge dz^2$ in $\mathbb{C}^2$). In that particular setting, minimal Lagrangian surfaces are calibrated, and thus minimizing; they are called special Lagrangian (see [Wo, HaL]).

## 1.2   Symmetric space structure

Let us now consider a Hermitian symmetric space of the form $M = G/H$, where $G$ is the group of unitary transformations of $M$ and $H$ the isotropy group at

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first vectors span the tangent space of $N$, i.e. they are in the same class of the frame bundle modulo $SO(n) \times SO(m-n)$. As much as Darboux framings give insight to the Riemannian structure, Lagrangian framings yield valuable information on the Lagrangian immersion. See [G] for a description of the moving frame theory.

[2] we differ from Wolfson [Wo] in that $\beta$ is defined mod $2\pi$ instead of mod 2.

some point henceforth denoted by $p_0$. The symmetry around $p_0$ is induced by a group involution $\sigma$ of $G$ with $(G^\sigma)_0 \subset H \subset G^\sigma$. Abusing notations, we will also write $\sigma$ for $d\sigma(e)$, its differential at identity, acting on $T_e G = \mathfrak{g}$, and similarly, thinking of elements of $G$ as matrices (which indeed they will be) we will identify $dg(p)$ with $g$.

The Lie algebra $\mathfrak{g}$ (of $G$) is the direct sum of two subspaces $\mathfrak{h} \oplus \mathfrak{m}$, eigenspaces of $\sigma$ with respective eigenvalue $+1$ and $-1$ (so that $\mathfrak{h}$ is a Lie subalgebra, the Lie algebra of $H$). The subspace $\mathfrak{m}$ identifies with $T_{p_0} M$. As a consequence, $\mathfrak{m}$ inherits a Hermitian structure. The Reader should consult [BRa] for a more thorough introduction to homogeneous spaces from a geometer's point of view. Fixing a unitary frame $(\epsilon_1, \epsilon_2)$ of $\mathfrak{m}$, any element $g \in G$ maps that reference frame at $p_0$ to another unitary frame $(e_1, e_2)$ at $p = g \cdot p_0$ through its differential $g (= dg(e))$.

Suppose now that we can lift the map $f : L \to M$ to $F : L \to G$. If, as will often the case, $L$ is a contractible domain, such lifts do exist. Thus a choice of $F$ yields a moving frame[3]. The choice is a priori wide and we have a gauge group $C^\infty(L, H)$ acting on lifts. Since we want the lift $F$ to encode some informations about the first derivatives of $f : L \to M$, we want to restrict our possibilities a little bit, and use only Lagrangian lifts, i.e. Lagrangian framings in the sense of definition 1.1. Lagrangian lifts do exist as will be obvious from the analysis below. Consider for the moment any lift $F$ and $\alpha = F^{-1}dF$ the associated (left) Maurer-Cartan form (the pull-back of the Maurer-Cartan form on $G$). Using the above symmetric splitting, we write $\alpha = \alpha_\mathfrak{h} + \alpha_\mathfrak{m}$; the $\mathfrak{m}$-valued 1-form $\alpha_\mathfrak{m}$ is the pull-back of the Maurer-Cartan form of the symmetric space $M$ (as defined in [BRa]); it allows us to identify tangent vectors to $L \subset M$ with elements of $\mathfrak{m}$. However this identification is gauge dependent: if we replace $F$ with $Fh^{-1}$ where $h \in C^\infty(L, H)$, $\alpha_\mathfrak{m}$ changes to $\mathrm{Ad}\, h(\alpha_\mathfrak{m})$. Using a reference unitary frame $(\epsilon_1, \epsilon_2)$, there exist at any $z \in L$ a unique element $a(z) \in GL(\mathfrak{m})$ such that

$$\alpha_\mathfrak{m}\big|_z = a(z)(\epsilon_1 dx + \epsilon_2 dy). \tag{1}$$

Without loss of generality, we may and will assume that $f$ is conformal (and $L$ endowed with the induced conformal structure making it a Riemann surface; $z = x + iy$ is always a local holomorphic coordinate). Since $(x, y)$ are conformal coordinates and $L$ is Lagrangian, (1) can be written as

$$\alpha_\mathfrak{m}\big|_z = e^{\rho(z)}k(z)(\epsilon_1 dx + \epsilon_2 dy) , \quad k(z) \in U(\mathfrak{m}) \tag{2}$$

where $e^{\rho(z)}$ is the conformal factor[4].

To understand the notion of Lagrangian lift we need to delve deeper into the structure of $G$ and $\mathfrak{g}$. First, as we shall see in later sections, there exists an order

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[3]beware that the reciprocal is not always true: one cannot always find a lift corresponding to a given moving frame; for instance it is true if $H$ is four dimensional (as in $\mathbb{CP}^2$) and false otherwise (as in $\mathbb{CP}^1 \times \mathbb{CP}^1$).

[4] If $\mathrm{Ad}\, H$ is big enough – namely transitive on orthonormal couples – then a suitable gauge change reduces $k(z)$ to the identity; such a lift is said *fundamental* (as for instance in $\mathbb{C}^2$, see [HR1]). However that notion is dependent on the coordinate $z$, unlike the notion of *Lagrangian lift* as will become clear hereafter.

four automorphism $\tau$ acting on $G$, squaring to $\tau^2 = \sigma$. Thus we may split $\mathfrak{g}^{\mathbb{C}}$ as $\mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_2^{\mathbb{C}} \oplus \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}}$, where $\mathfrak{g}_a^{\mathbb{C}}$ is the $i^a$-eigenspace of $\tau$. Since $\tau^2 = \sigma$, $\mathfrak{h}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_2^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}}$. Let $G_0$ be the subgroup of $G$ fixed by $\tau$, whose Lie algebra is of course $\mathfrak{g}_0$. Recall that the adjoint action of $H$ on $\mathfrak{m}$ identifies with the action of the linear isotropy $H^* = \{dh(p_0), h \in H\}$ on $T_{p_0}M$ (see [GaHuLa], Chapter 1); in particular, since all elements of $G$ are unitary transformations of $M$, $\operatorname{Ad} H$ identifies with a subgroup of $U(\mathfrak{m})$ (and $\operatorname{ad}\mathfrak{h}$ with a subalgebra of $\mathfrak{u}(\mathfrak{m})$). Moreover, as we shall see later on, there always is an element $Y \in \mathfrak{h}$ such that $\exp(\frac{\pi}{2}\operatorname{ad} Y) = \operatorname{Ad}\exp(\frac{\pi}{2}Y)$ is the complex structure $J$ on $\mathfrak{m}$ and the automorphism $\tau$ can be chosen so that $\mathfrak{g}_2 = \mathbb{R}Y$ (and $\operatorname{Ad} G_0 \subset SU(\mathfrak{m})$).

In order to picture the moving frames, remember that the tangent bundle $TN$ is canonically diffeomorphic to the subbundle $[\mathfrak{m}]$ of $N \times \mathfrak{g}$ with fiber $\operatorname{Ad} g(\mathfrak{m})$ over the point $g \cdot p_0$. The moving frame induced by $F$ is $(\operatorname{Ad} F(\epsilon_1), \operatorname{Ad} F(\epsilon_2))$; and it is clear now that two lifts $F, F'$ of $f$ define the same moving frame mod $SU(2)$ if and only if they are gauge equivalent modulo the restricted gauge group $C^\infty(L, G_0)$. In these local coordinates a unitary framing $(e_1, e_2) \simeq \left( e^{-\rho} \frac{\partial f}{\partial x}, e^{-\rho} \frac{\partial f}{\partial y} \right)$ can be identified (in $[\mathfrak{m}]$) with

$$\operatorname{Ad} F \left( e^{-\rho} \alpha_{\mathfrak{m}} \left( \frac{\partial}{\partial x} \right), e^{-\rho} \alpha_{\mathfrak{m}} \left( \frac{\partial}{\partial y} \right) \right) \simeq \operatorname{Ad} F(z) \left( k(z)\epsilon_1, k(z)\epsilon_2 \right)$$

using (2) – the later expressions being (inconspicuously) independent of the choice of the lift $F$. The tangent section is the class mod $SU(2)$ of $(e_1, e_2)$. The lift $F$ is Lagrangian if and only if its induced moving frame lies in the same class as $(e_1, e_2)$. That requires exactly that $k$ be special unitary ($\det_{\mathbb{C}} k(z) = 1$). That is a very simple condition to check. Such lifts do exist on contractible domains since a gauge change $F \to F\exp(-\theta Y)$ multiplies $\det_{\mathbb{C}} k$ by $\det_{\mathbb{C}} \operatorname{Ad}\exp(\theta Y) = e^{2i\theta}$; we may thus choose $\theta$ to force $k \in SU(\mathfrak{m})$. Notice that

(i) no integrability condition is involved in this process,

(ii) $\theta$ is defined mod $\pi$, so that the process can be applied to non simply-connected domains, by taking a suitable multiple covering (namely double covering for each generator).

How does one read the Lagrangian angle now ? First we have to build a parallel lift $\bar{F}$; recall that the covariant derivative is simply the flat differentiation followed by the projection on the bundle $[\mathfrak{m}]$, that we denote here by $\lfloor \ldots \rfloor_{[\mathfrak{m}]}$, then we have for $i = 1, 2$ (denoting $\bar{\alpha} = \bar{F}^{-1} d\bar{F}$)

$$\begin{aligned}
\nabla \bar{e}_i &= \left\lfloor d(\operatorname{Ad}\bar{F}(\epsilon_i)) \right\rfloor_{[\mathfrak{m}]} = \left\lfloor \operatorname{Ad}\bar{F}([\bar{\alpha}, \epsilon_i]) \right\rfloor_{[\mathfrak{m}]} = \operatorname{Ad}\bar{F}(\lfloor ([\bar{\alpha}, \epsilon_i]) \rfloor_{\mathfrak{m}}) \\
&= \operatorname{Ad}\bar{F}([\bar{\alpha}_{\mathfrak{h}}, \epsilon_i]) = [\operatorname{Ad}\bar{F}(\bar{\alpha}_{\mathfrak{h}}), \bar{e}_i]
\end{aligned}$$

so that the unitary frame $\bar{s} := (\bar{e}_1, \bar{e}_2)$ varies according to $\nabla \bar{s} = \zeta(s)$ with $\zeta : V \mapsto [\operatorname{Ad}\bar{F}(\bar{\alpha}_{\mathfrak{h}}), V]$. The class modulo $SU(2)$ of $s$ varies following the class modulo $\mathfrak{su}(2)$ of $\zeta$, i.e. it is the projection of $\zeta$ on the diagonal (central)

part $\mathfrak{z}(\mathrm{Ad}\,\bar{F}(\mathfrak{m}))$ in the Lie algebra decomposition $\mathfrak{u}(\mathrm{Ad}\,\bar{F}(\mathfrak{m})) = \mathfrak{z}(\mathrm{Ad}\,\bar{F}(\mathfrak{m})) \oplus \mathfrak{su}(\mathrm{Ad}\,\bar{F}(\mathfrak{m}))$. Since

$$\lfloor\zeta\rfloor_{\mathfrak{z}(\mathrm{Ad}\,\bar{F}(\mathfrak{m}))} = \mathrm{Ad}\,\bar{F}\left\lfloor\mathrm{Ad}\,\bar{F}^{-1}\zeta\,\mathrm{Ad}\,\bar{F}\right\rfloor_{\mathfrak{z}(\mathfrak{m})} \mathrm{Ad}\,\bar{F}^{-1} = \mathrm{Ad}\,\bar{F}\left\lfloor\mathrm{ad}\,\bar{\alpha}_\mathfrak{h}\right\rfloor_{\mathfrak{z}(\mathfrak{m})} \mathrm{Ad}\,\bar{F}^{-1},$$

$\bar{F}$ is flat if and only if $\left\lfloor\mathrm{ad}\,\bar{\alpha}_\mathfrak{h}\right\rfloor_{\mathfrak{z}(\mathfrak{m})} = 0$, that is $\bar{\alpha}_2 = 0$, because the adjoint representation maps $\mathfrak{g}_2$ to $\mathfrak{z}(\mathfrak{m})$ and $\mathfrak{g}_0$ into $\mathfrak{su}(\mathfrak{m})$. We now have four ways of reading the Lagrangian angle:

1. $e^{i\beta}$ is the complex determinant of the tangent frame $(\mathrm{Ad}\,\bar{F}(z)(\bar{k}(z)\epsilon_1)$, $\mathrm{Ad}\,\bar{F}(z)(\bar{k}(z)\epsilon_2))$ in the parallel frame $(\mathrm{Ad}\,\bar{F}(z)(\epsilon_1), \mathrm{Ad}\,\bar{F}(z)(\epsilon_2))$, namely is the complex determinant of $\bar{k}$ for any parallel framing of $f^*B'$,

2. given a Lagrangian lift $F$, and $h = \bar{F}^{-1}F$ the ($H$-valued) gauge change, then $e^{i\beta} = \det_\mathbb{C}\mathrm{Ad}\,h$,

3. in the previous gauge change, write $h$ as a (commutative) product $h_2 h_0$ in $G_0 G_2$, which is unique up to sign ($G_2 = \exp\mathfrak{g}_2$); then $h_2 = \exp(\beta Y/2)$,

4. finally, for a Lagrangian lift, it follows from the previous characterization (3) that

$$\begin{aligned}\frac{d\beta}{2}Y &= h_2^{-1}dh_2 = \left\lfloor\mathrm{Ad}\,h_0(h^{-1}dh) - dh_0 h_0^{-1}\right\rfloor_{\mathfrak{g}_2} \\ &= \left\lfloor h^{-1}dh\right\rfloor_{\mathfrak{g}_2} = \left\lfloor\alpha - \mathrm{Ad}\,F^{-1}(\bar{\alpha})\right\rfloor_{\mathfrak{g}_2} = \alpha_2.\end{aligned}$$

In the following we shall actually exploit the last characterization. We conclude by noting that the Maslov form is exactly $\frac{2}{\pi}$ times the $Y$ component of $\alpha_2$.

So far we have not used at all the underlying complex structure of $L$. But decomposing the Maurer-Cartan form $\alpha$ into its $(1,0)$ and $(0,1)$ parts $\alpha'$ and $\alpha''$ respectively, yields an interesting condition for a lift $F$ to be Lagrangian:

**Proposition 1.3** *A lift is Lagrangian if and only if the Maurer-Cartan form $\alpha = \alpha_2 + \alpha_0 + \alpha_{-1} + \alpha_1$ satisfies $\alpha''_{-1} = 0$ (which by reality assumption implies $\alpha'_1 = 0$).*

*Proof.* The equivalence rests upon the following simple fact: $\mathfrak{g}^\mathbb{C}_{-1}$ is *exactly* the orbit under $\mathbb{R}_+ \times SU(\mathfrak{m})$ of the vector $\epsilon = \frac{1}{2}(\epsilon_1 - i\epsilon_2) \in \mathfrak{m}^\mathbb{C}$ (for any choice of Hermitian basis $(\epsilon_1, \epsilon_2)$). A proof is given in [HR1]. Now writing $\alpha_\mathfrak{m} = e^\rho k(\epsilon dz + \bar{\epsilon}d\bar{z})$, we see that $\alpha'_\mathfrak{m} = e^\rho k\epsilon dz$ belongs to $\mathfrak{g}^\mathbb{C}_{-1}$ if and only if $k$ is in $SU(\mathfrak{m})$. ∎

This type of condition should be compared with *primitivity* conditions [BP] or $\omega$-maps [Hi]. However our "partial primitivity" differs in that it is a first order requirement on the immersion (namely that of being Lagrangian) and not an Euler-Lagrange condition, while primitive maps and $\omega$-maps are automatically harmonic. Finally we remark that the assumption of conformality is not crucial until the very final step; indeed the concept of Lagrangian lift makes sense in any dimension.

## 1.3 Lagrangian surfaces in $\mathbb{CP}^2$ and Lagrangian cones in $\mathbb{C}^3$

We concern ourselves here with Lagrangian three dimensional cones centered at the origin in $\mathbb{C}^3$, and in particular with their intersection with the unit sphere $S^5$, which is called the *link*. We assume henceforward that the cone is regular, i.e. the link $M$ is a connected submanifold. Notice that our analysis applies as well to regular conical singularities in complex three dimensional manifolds. We recall here the known correspondence between Lagrangian cones and Lagrangian surfaces in $\mathbb{CP}^2$ (see for instance [Re1, Re2]).

Lagrangian cones in $\mathbb{C}^3$ may be locally described by Lagrangian surfaces in $\mathbb{CP}^2$. This is done by the canonical projection map $P : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$. More precisely to any Lagrangian cone $\Sigma^3$ in $\mathbb{C}^3$ we may associate the Lagrangian surface $L := P(\Sigma^3)$ in $\mathbb{C}^3$. Conversely this surface $L$ describes completely $\Sigma^3$ up to a rigid motion in the sense that the set of Lagrangian cones of $\mathbb{C}^3$ which are mapped to $L$ by $P$ is $\{e^{i\alpha}\Sigma^3/\alpha \in \mathbb{R}/2\pi\mathbb{Z}\}$. This follows from the following picture. Let $\langle .,.\rangle_H = \langle .,.\rangle_E - i\omega(.,.)$ be the standard Hermitian product in $\mathbb{C}^3$ ($\langle .,.\rangle_E$ is the Euclidean scalar product and $\omega(.,.)$ the symplectic form). Let

$$S^5 := \{Z \in \mathbb{C}^3/\langle Z, Z\rangle_H = 1\}$$

be the unit sphere in $\mathbb{C}^3$, and $\mathcal{L} := \Sigma \cap S^5$ its link, which fully determines $\Sigma$. Now at any point $p \in \mathcal{L} \subset \Sigma^3$ choose an orthonormal frame $(e_1, e_2, e_3)$ of $T_p\Sigma^3$ such that $e_3 = p$. Then $(e_1, e_2)$ is an orthonormal basis of $T_p\mathcal{L}$. The Lagrangian constraint on $\Sigma$ at the point $p$ can be stated as $iT_pM \perp T_pM$; this is equivalent to the two conditions

**a)** $T_p\mathcal{L} \perp ip$ and

**b)** $ie_2 \perp e_1$.

The first condition **a)** means that $T_p\mathcal{L}$ is contained in $\Pi_p$, the 4-dimensional subspace of $T_pS^5$ orthogonal to $ip$. The collection $\Pi := (\Pi_p)_{p \in S^5}$ forms a distribution on $S^5$, orthogonal to the fibers of the Hopf fibration $\mathcal{H} : S^5 \to \mathbb{CP}^2$, and therefore named the *horizontal* distribution. Notice that each $\Pi_p$ can also be identified with the orthogonal subspace to $p$ in $\mathbb{C}^3$ for the Hermitian product. The restriction of $\langle .,.\rangle_H$ to $\Pi_p$ is also Hermitian, which implies that the restriction of $\omega$ to $\Pi_p$ is symplectic (non degenerate). Now the second condition **b)** just means that $T_p\mathcal{L}$ is a Lagrangian plane in $\Pi_p$. These two conditions actually ensure that $\mathcal{L}$ is a *Legendrian* submanifold of the contact manifold $(S^5, \Pi, \omega_{|\Pi})$.

The horizontality condition **a)** on $\mathcal{L}$, $T_p\mathcal{L} \subset \Pi_p$, $\forall p$ can be translated into the property that the restriction of the Hopf fibration $\mathcal{H}$ to $\mathcal{L}$ is an isometric covering map (and locally an isometric diffeomorphism onto its image). The second condition **b)** is then contained in the property that $\mathcal{H}(\mathcal{L})$ is a Lagrangian submanifold of $\mathbb{CP}^2$.

Conversely let us start from a contractible Lagrangian $L \subset \mathbb{CP}^2$ and let us try to construct first a Legendrian link lifting $\mathcal{L}$ in $S^5$ and then a Lagrangian

cone in $\mathbb{C}^3$. We use the canonical connection $\nabla$ on the Hopf fibration defined for any section $s : \mathbb{CP}^2 \to S^5$, any point $\underline{p} \in \mathbb{CP}^2$ and any $X \in T_{\underline{p}}\mathbb{CP}^2$ by

$$\nabla_X s(\underline{p}) := \langle ds_{\underline{p}}(X), s(\underline{p})\rangle_H s(\underline{p}).$$

The curvature of $\nabla$ is given by $\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s = 2is^\star\omega_{\mathbb{C}^3}(X,Y)s$ and hence vanishes along any Lagrangian surface. It follows that, if $L$ is Lagrangian and if $f$ denotes the immersion mapping $L \subset \mathbb{CP}^2$, then $f^\star\mathcal{H}$, the pull-back of the Hopf bundle by $f$ is flat. Since $L$ is also contractible we can construct a flat section $s : L \to S^5$ (unique up to the choice of the value of $s$ at one point) and $\mathcal{L} := s(L)$ is just the Legendrian lift of $L$ that we were looking for. Now this Legendrian surface spans a Lagrangian cone $\Sigma^3$ in $\mathbb{C}^3$.

Furthermore we observe that $\Sigma^3$ is Hamiltonian stationary if and only if $L$ is so. We denote $\Theta := dz^1 \wedge dz^2 \wedge dz^3$ the 3-form which helps us to define the Lagrangian angle function on $\Sigma^3$ by $e^{i\beta} = \Theta(e_1, e_2, e_3)$ for any orthonormal basis $(e_1, e_2, e_3)$ of $T_p\Sigma^3$. The cone $\Sigma^3$ is Hamiltonian stationary if and only if $\Delta_{\Sigma^3}\beta = 0$. Since $\beta$ obviously does not depend on the radius $r := \langle p, p\rangle_H^{1/2}$, this condition is equivalent to $\Delta_\mathcal{L}\beta = 0$ along $\mathcal{L}$. Now we need a parallel section $\theta$ of $f^\star K$, where $K$ is the canonical bundle of $\mathbb{CP}^2$ and $f$ denotes the immersion $L \subset \mathbb{CP}^2$. A very simple construction is the following: let $s : L \to \mathcal{L}$ be the lift mapping, it is a parallel section of $f^\star\mathcal{H}$ - because $\mathcal{L}$ is Legendrian - and hence the 2-form $\theta := s^\star(\iota_s\Theta)$ is parallel. We can thus use $\theta$ to construct the Lagrangian angle along $L$. Let us denote $\underline{p}$ a point in $L$ and $p = s(\underline{p})$ its lift in $\mathcal{L}$. The Lagrangian angle $\underline{\beta}$ at $\underline{p}$ is defined by $e^{i\underline{\beta}} = \theta(\underline{e}_1, \underline{e}_2)$, where $(\underline{e}_1, \underline{e}_2)$ is an orthonormal basis of $T_{\underline{p}}L$. But since $(p, s_\star\underline{e}_1, s_\star\underline{e}_2)$ is an orthonormal basis of $T_p\Sigma^3$, $\theta(\underline{e}_1, \underline{e}_2) = \Theta(s(\underline{p}), s_\star\underline{e}_1, s_\star\underline{e}_2) = e^{i\beta}$ and hence the two Lagrangian angles coincide. It follows that the harmonicity condition on $\beta$ along $\mathcal{L}$ is equivalent to the harmonicity condition on $\underline{\beta}$ along $L$, i.e. the condition that $L$ is Hamiltonian stationary.

Another presentation of this relationship between Lagrangian surfaces in $\mathbb{CP}^2$ and Lagrangian cones in $\mathbb{C}^3$ from the moving frame point of view will be found in §4.3.

## 2 Loop group formulation

### 2.1 A general formulation of the problem using a family of curvature free connections

In order to yield a well-defined Lagrangian immersion $f$ on the contractible domain $L$, the Maurer-Cartan form $\alpha$ of a Lagrangian lift $F$ needs only to satisfy the closedness condition (also called zero curvature equation, when thinking of $d + \alpha$ as bundle connection): $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$. This equation splits along the

eigenspaces to give:

$$\begin{cases} d\alpha_2 + [\alpha_0 \wedge \alpha_2] + \frac{1}{2}[\alpha_{-1} \wedge \alpha_{-1}] + \frac{1}{2}[\alpha_1 \wedge \alpha_1] = 0 \\ d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] + \frac{1}{2}[\alpha_2 \wedge \alpha_2] + [\alpha_{-1} \wedge \alpha_1] = 0 \\ d\alpha_{-1} + [\alpha_0 \wedge \alpha_{-1}] + [\alpha_2 \wedge \alpha_1] = 0 \\ d\alpha_1 + [\alpha_0 \wedge \alpha_1] + [\alpha_2 \wedge \alpha_{-1}] = 0 \end{cases}$$

However due to the commutation relations $[\mathfrak{g}_2, \mathfrak{g}_2] = [\mathfrak{g}_2, \mathfrak{g}_0] = 0$, and to the properties of our Lagrangian lift, we simplify that to:

$$\begin{cases} d\alpha_2 = 0 \\ d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] + [\alpha'_{-1} \wedge \alpha''_1] = 0 \\ d\alpha'_{-1} + [\alpha_0 \wedge \alpha'_{-1}] + [\alpha_2 \wedge \alpha''_1] = 0 \\ d\alpha''_1 + [\alpha_0 \wedge \alpha''_1] + [\alpha_2 \wedge \alpha'_{-1}] = 0 \end{cases} \tag{3}$$

As noted in [CM, Wo], a Lagrangian surface is Hamiltonian stationary, i.e. stationary with respect to Hamiltonian deformations, if and only if $\beta$ is harmonic, that is $\Theta$ is coclosed, in other words $d \star \alpha_2 = 0$. Furthermore this Lagrangian surface is minimal[5] if and only if $\alpha_2 = 0$. A now classical trick allows us to join these two differential equations into one zero curvature equation, though formulated on a loop algebra.

**Proposition 2.1** *On a simply connected domain $L$, the $\mathfrak{g}$-valued one-form $\alpha$ is the Maurer-Cartan of a weakly conformal Lagrangian immersion if and only if it is flat: $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ and partially primitive: $\alpha''_{-1} = \alpha'_1 = 0$. Furthermore the immersion is Hamiltonian stationary if and only if the* extended Maurer-Cartan form $\alpha_\lambda = \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1 + \lambda^2\alpha''_2$ *is flat for all $\lambda \in \mathbb{C}^*$ (or $S^1$):*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \tag{4}$$

*with the minimal[5] subcase characterized by $\alpha_2 = 0$.*

*Proof.* Just check that the only additional condition induced by (4) is the coclosedness condition on $\alpha_2$. ∎

Equation (4) is a curvature free condition, a.k.a. the compatibility condition for the existence for any $\lambda \in S^1$ of a solution $F_\lambda : L \to G$ of the equation

$$dF_\lambda = F_\lambda \alpha_\lambda. \tag{5}$$

Moreover $F_\lambda$ is unique provided that we know its value at some point $p_0 \in L$. We may choose for instance $F_\lambda(p_0) = \mathbb{1}$. Is is not difficult to realize that each $F_\lambda$ is a lift of a Hamiltonian stationary Lagrangian surface like $F$. We deduce the following:

**Corollary 2.2** *Local Hamiltonian stationary Lagrangian surfaces come in families parametrized by $\lambda \in S^1$. For global surfaces there usually are period problems.*

---

[5]special Lagrangian if $M = \mathbb{C}^2$.

## 2.2 Loop groups

In the light of Proposition 2.1 and Corollary 2.2 above it is natural to introduce the following loop groups - following now classical techniques since [SW] (based on ideas which come back to [SaSa]) and [U]. We define the set of maps from $S^1$ to $G$:

$$\Lambda G := \{S^1 \ni \lambda \mapsto \varphi_\lambda \in G\}.$$

We assume that these maps are bounded in the $H^s$ topology, with $s > 1/2$, where using the Fourier decomposition $\varphi_\lambda = \sum_{k \in \mathbb{Z}} \hat{\varphi}_k \lambda^k$, the $H^s$ norm is defined by $||\varphi_\lambda||_s^2 := \sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{\varphi}_k|^2$. Then $\Lambda G$ is a group for the composition law $[\lambda \mapsto \varphi_\lambda].[\lambda \mapsto \psi_\lambda] = [\lambda \mapsto \varphi_\lambda \psi_\lambda]$. We also consider the twisted loop (sub)group

$$\Lambda G_\tau := \{S^1 \ni \lambda \mapsto \varphi_\lambda \in G / \tau(\varphi_\lambda) = \varphi_{i\lambda}\}.$$

These loop groups have Lie algebras which are respectively

$$\Lambda\mathfrak{g} := \{S^1 \ni \lambda \mapsto \xi_\lambda \in \mathfrak{g}\}$$

and

$$\Lambda\mathfrak{g}_\tau := \{S^1 \ni \lambda \mapsto \xi_\lambda \in \mathfrak{g} / \tau(\xi_\lambda) = \xi_{i\lambda}\}.$$

A key observation is that $\alpha_\lambda$ can be seen as a 1-form with values in $\Lambda\mathfrak{g}_\tau$. More precisely, "partially primitive" extended 1-forms are exactly the 1-forms $\alpha_\lambda$ with values in $\Lambda\mathfrak{g}_\tau$ such that $\lambda^2 \alpha_\lambda$ has a limit when $\lambda$ goes to zero. Similarly, choosing $F_\lambda(p_0) = \mathbb{1}$, the family of maps $F_\lambda : L \to G$ solution of (5) can rather be viewed as a map into $\Lambda G_\tau$, called the *extended lift* of $f$. Such loop groups have already been considered in [BP, DPW] in the context of harmonic maps into a symmetric space or in [HR1] for Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$.

# 3 Weierstrass type representations

As in [HR1], the above loop formulation of the Hamiltonian stationary Lagrangian surface problem opens the gate to the use of various constructions of solutions to this problem, using completely integrable systems. As an illustration, we will present here Weierstrass representations in the spirit of J. Dorfmeister, F. Pedit and H.Y. Wu [DPW].

## 3.1 Loop groups decompositions

At the base of this construction is the idea of Iwasawa decomposition. For instance we shall need such a property for $G_0$, the subgroup of $G$ fixed by $\tau$, namely: there exists a solvable Lie subgroup $B_{G_0}$ of $G_0^\mathbb{C}$ such that the following mapping

$$\begin{array}{ccc} G_0 \times B_{G_0} & \longrightarrow & G_0^\mathbb{C} \\ (g, b) & \longmapsto & gb \end{array}$$

is a diffeomorphism. We summarize by $G_0^{\mathbb{C}} = G_0 B_{G_0}$ this property is named *Iwasawa decomposition*. But we actually need more: an infinite dimensional extension of this property to loop groups.

For that purpose we need to introduce the complexified versions of the above loop groups, obtained by replacing $G$ by its complexification $G^{\mathbb{C}}$:

$$\Lambda G^{\mathbb{C}} := \{S^1 \ni \lambda \mapsto \varphi_\lambda \in G^{\mathbb{C}}\},$$

$$\Lambda G_\tau^{\mathbb{C}} := \{S^1 \ni \lambda \mapsto \varphi_\lambda \in G^{\mathbb{C}} / \tau(\varphi_\lambda) = \varphi_{i\lambda}\}$$

and their Lie algebras $\Lambda \mathfrak{g}^{\mathbb{C}}$ and $\Lambda \mathfrak{g}_\tau^{\mathbb{C}}$. And we also introduce the subgroups

$$\Lambda^+ G_\tau^{\mathbb{C}} := \{[\lambda \mapsto \varphi_\lambda] \in \Lambda G_\tau^{\mathbb{C}} \text{ extending holomorphically in the disk } D^2\},$$

$$\Lambda_{B_{G_0}}^+ G_\tau^{\mathbb{C}} := \{[\lambda \mapsto \varphi_\lambda] \in L^+ G_\tau^{\mathbb{C}} / \varphi_0 \in B_{G_0}\},$$

$$\Lambda_\star^- G_\tau^{\mathbb{C}} := \{[\lambda \mapsto \varphi_\lambda] \in \Lambda G_\tau^{\mathbb{C}} \text{ extending holomorphically in } S^2 \backslash D^2 \text{ and } \varphi_\infty = 1\}.$$

The two main tools are the following Lemmas, which are proved in [DPW] (the proofs are based on Theorems in [PrS]).

**Lemma 3.1** *Assume that $G$ is a compact Lie group. Let $\tau : G \to G$ be an order four automorphism of $G$ and let $G_0$ be the subgroup of $G$ fixed by $\tau$. Suppose that the Iwasawa decomposition $G_0^{\mathbb{C}} = G_0 B_{G_0}$ holds. Then the mapping*

$$
\begin{array}{ccc}
\Lambda G_\tau \times \Lambda_{B_{G_0}}^+ G_\tau^{\mathbb{C}} & \longrightarrow & \Lambda G_\tau^{\mathbb{C}} \\
(g_\lambda, b_\lambda) & \longmapsto & g_\lambda b_\lambda
\end{array}
$$

*is a diffeomorphism. We denote by $\Lambda G_\tau^{\mathbb{C}} = \Lambda G_\tau . \Lambda_{B_{G_0}}^+ G_\tau^{\mathbb{C}}$ this property.*

**Lemma 3.2** *Assume that $G$ is a semisimple Lie group. Then there exists a dense open subset $\mathcal{C}$ of the connected component of the identity of $\Lambda G_\tau^{\mathbb{C}}$, called the* big cell, *such that the mapping*

$$
\begin{array}{ccc}
\Lambda_\star^- G_\tau^{\mathbb{C}} \times \Lambda^+ G_\tau^{\mathbb{C}} & \longrightarrow & \mathcal{C} \\
(\varphi_\lambda^-, \varphi_\lambda^+) & \longmapsto & \varphi_\lambda^- \varphi_\lambda^+
\end{array}
$$

*is a diffeomorphism. We denote by $\mathcal{C} = \Lambda_\star^- G_\tau^{\mathbb{C}} . \Lambda^+ G_\tau^{\mathbb{C}}$ this property.*

In some cases in this paper these results do not apply directly, either because the isometry group $G$ is not compact (for $\mathbb{C}^2$, $\mathbb{CD}^2$ or the dual of $\mathbb{CP}^1 \times \mathbb{CP}^1$) or because this group is not semisimple (in the case of $\mathbb{C}^2$). However it is possible to extend the above Lemmas to these situations in two ways:

- for $G = U(2) \ltimes \mathbb{C}^2$, the properties stated in Lemmas 3.1 and 3.2 are true; it was proved in [HR1] by a direct construction.

- in all cases, in particular when $G$ is not compact or not semi-simple, local versions of Lemmas 3.1 and 3.2 can be proven. In these versions, one just need to replace the loop groups by a neighborhood of the identity. The proof of these results uses the inverse mapping theorem as in [H].

## 3.2   Solutions in terms of holomorphic data

Conformal immersions of Hamiltonian stationary Lagrangian surfaces are in correspondence with holomorphic data as defined below. We first denote

$$\Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}} := \{[\lambda \mapsto \xi_\lambda] \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}/\xi_\lambda = \sum_{k=-2}^{\infty} \hat{\xi}_k\lambda^k\}.$$

**Definition 3.3** *The set of* holomorphic potentials, *denoted* $\mathcal{H}_{-2,\infty}(L)$, *is the set of holomorphic 1-forms on $L$ with values in $L_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$. So any form $\mu_\lambda$ in $\mathcal{H}_{-2,\infty}(L)$ has the expression*

$$\mu_\lambda = \sum_{k=-2}^{\infty} \hat{\mu}_k\lambda^k = \sum_{k=-2}^{\infty} \hat{\xi}_k(z)\lambda^k dz,$$

*where* $\forall z,\ \sum_{k=-2}^{\infty} \hat{\xi}_k(z)\lambda^k \in \Lambda_{-2,\infty}\mathfrak{g}_\tau^{\mathbb{C}}$.

**Lemma 3.4** *Let $F_\lambda : L \to \Lambda G_\tau$ be the extended lift of a (conformal) Hamiltonian stationary Lagrangian immersion and assume that $L$ is contractible. Then*

- *there exist a holomorphic map $H_\lambda : L \to \Lambda G_\tau^{\mathbb{C}}$ and a map $B_\lambda : L \to \Lambda_{B_{G_0}}^+ G_\tau^{\mathbb{C}}$ such that $F_\lambda = H_\lambda B_\lambda$.*

- *the Maurer-Cartan form $\mu_\lambda := (H_\lambda)^{-1}dH_\lambda$ is a holomorphic potential.*

*Proof.*   (see [DPW] for details) The existence of $H_\lambda$ and $B_\lambda$ relies on solving the equation

$$0 = \frac{\partial(F_\lambda(B_\lambda)^{-1})}{\partial\overline{z}} = F_\lambda\left(\alpha_\lambda\left(\frac{\partial}{\partial\overline{z}}\right) - (B_\lambda)^{-1}\frac{\partial B_\lambda}{\partial\overline{z}}\right)(B_\lambda)^{-1},$$

which is equivalent to

$$\frac{\partial B_\lambda}{\partial\overline{z}} = B_\lambda(\alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2)\left(\frac{\partial}{\partial\overline{z}}\right),$$

with the constraint that $B_\lambda$ takes values in $L_{B_{G_0}}^+ G_\tau^{\mathbb{C}}$. The existence of a solution is first obtained locally, then we can glue local solutions into a global one. This proves the first assertion. Now we write

$$(H_\lambda)^{-1}dH_\lambda = B_\lambda(\alpha_\lambda - (B_\lambda)^{-1}dB_\lambda)(B_\lambda)^{-1},$$

and using the fact that $B_\lambda$ takes values in $L_{B_{G_0}}^+ G_\tau^{\mathbb{C}}$ and that $z \mapsto H_\lambda(z)$ is holomorphic, we deduce that $\mu_\lambda := (H_\lambda)^{-1}dH_\lambda$ has the desired properties.   ∎

Conversely any holomorphic potential in $\mathcal{H}_{-2,\infty}(L)$ produces a Hamiltonian stationary Lagrangian immersion as follows.

**Theorem 3.5** *Let $\mu_\lambda \in \mathcal{H}_{-2,\infty}(L)$, $p_0$ a point in $L$ and $H_\lambda^0$ a constant in $\Lambda G_\tau^\mathbb{C}$. Then*

- *there exists a unique holomorphic map $H_\lambda : L \to \Lambda G_\tau^\mathbb{C}$, such that $dH_\lambda = H_\lambda \mu_\lambda$ and $H_\lambda(p_0) = H_\lambda^0$.*

- *if the loop groups decomposition $\Lambda G_\tau^\mathbb{C} = \Lambda G_\tau . L_{B_{G_0}}^+ G_\tau^\mathbb{C}$ holds then we can apply it to $H_\lambda(z)$ for all value of $z$. It follows that there exists two maps $F_\lambda : L \to \Lambda G_\tau$ and $B_\lambda : L \to L_{B_{G_0}}^+ G_\tau^\mathbb{C}$ such that*

$$H_\lambda(z) = F_\lambda(z)B_\lambda(z), \quad \forall z \in L.$$

*Then $F_\lambda$ is a lift of a (conformal) Hamiltonian stationary Lagrangian immersion.*

*Proof.* Since $\mu_\lambda = \xi_\lambda dz$, with $\frac{\partial \xi_\lambda}{\partial \bar{z}} = 0$, it follows easily that $d\mu_\lambda + \mu_\lambda \wedge \mu_\lambda = 0$, hence the existence and the uniqueness of $H_\lambda$. Assume now that we can perform the generalized Iwasawa decomposition $H_\lambda = F_\lambda B_\lambda$. It implies that

$$(F_\lambda)^{-1}dF_\lambda = B_\lambda \mu_\lambda (B_\lambda)^{-1} - dB_\lambda (B_\lambda)^{-1}. \tag{6}$$

Now using the fact that $\mu_\lambda \in \mathcal{H}_{-2,\infty}(L)$ and $B_\lambda$ takes value in $\Lambda_{B_{G_0}}^+ G_\tau^\mathbb{C}$, it is easy to check that the right hand side of (6) has the form $\sum_{k=-2}^\infty \hat{\alpha}_k \lambda^k$. But (6) implies also that this quantity should be real, i.e. a 1-form with coefficients in $\Lambda G_\tau$. Hence $\alpha_\lambda := (F_\lambda)^{-1}dF_\lambda$ reduces to $\alpha_\lambda = \hat{\alpha}_{-2}\lambda^{-2} + \hat{\alpha}_{-1}\lambda^{-1} + \hat{\alpha}_0 + \hat{\alpha}_1 \lambda + \hat{\alpha}_2 \lambda^2$ and moreover $\hat{\alpha}_0$ is real, $\hat{\alpha}_1 = \overline{\hat{\alpha}_{-1}}$ and $\hat{\alpha}_2 = \overline{\hat{\alpha}_{-2}}$. Lastly a Taylor expansion in $\lambda$ of (6) proves that $\hat{\alpha}_{-2}$ and $\hat{\alpha}_{-1}$ are (1,0)-forms, which ensures the result by Proposition 2.1. ∎

## 3.3 Meromorphic potentials

The holomorphic potentials constructed in Lemma 3.4 are far from being unique. Moreover they involved in general infinitely many holomorphic maps. These defects can be mended, provided we allow meromorphic potentials and under some hypotheses on $G$. We define

$$L_{-2,-1}\mathfrak{g}_\tau^\mathbb{C} := \{[\lambda \mapsto \varphi_\lambda] \in \Lambda \mathfrak{g}_\tau^\mathbb{C}/\xi_\lambda = \hat{\xi}_{-2}\lambda^{-2} + \hat{\xi}_{-1}\lambda^{-1}\}.$$

**Definition 3.6** *The set of meromorphic potentials, denoted $\mathcal{M}_{-2,-1}(L)$, is the set of meromorphic 1-forms on $L$ with coefficients in $L_{-2,-1}\mathfrak{g}_\tau^\mathbb{C}$. So any form $\mu_\lambda$ in $\mathcal{M}_{-2,-1}(L)$ has the expression*

$$\mu_\lambda = \hat{\mu}_{-2}\lambda^{-2} + \hat{\mu}_{-2}\lambda^{-2} = (\hat{\xi}_{-2}(z)\lambda^{-2} + \hat{\xi}_{-1}(z)\lambda^{-1})dz,$$

*where $\hat{\xi}_{-2}(z)\lambda^{-2} + \hat{\xi}_{-1}(z)\lambda^{-1} \in L_{-2,\infty}\mathfrak{g}_\tau^\mathbb{C}$.*

Then using the same methods as in [DPW]; one can prove the following

**Theorem 3.7** *Assume that the conclusion of Lemma 3.2 holds. Let $F_\lambda : L \to \Lambda G_\tau$ be the extended lift of a (conformal) Hamiltonian stationary Lagrangian immersion. Then there exists a finite subset $\{a_1, \ldots, a_p\}$ of $L$ such that*

- *there exists a holomorphic map $F_\lambda^- : L \setminus \{a_1, \ldots, a_p\} \to \Lambda_\star^- G_\tau^\mathbb{C}$ and a map $F_\lambda^+ : L \setminus \{a_1, \ldots, a_p\} \to \Lambda^+ G_\tau^\mathbb{C}$ such that*

$$F_\lambda(z) = F_\lambda^-(z)F_\lambda^+(z), \quad \forall z \in L \setminus \{a_1, \ldots, a_p\}$$

- *$z \mapsto F_\lambda^-(z)$ extends to a meromorphic map on $L$*

- *the Maurer-Cartan form $\mu_\lambda := (F_\lambda^-)^{-1}dF_\lambda^-$ of $F_\lambda^-$ is a meromorphic potential in $\mathcal{M}_{-2,-1}(L)$.*

*Proof.* (see [DPW] for details) The decomposition $F_\lambda(z) = F_\lambda^-(z)F_\lambda^+(z)$ is possible as soon as we can prove that $F_\lambda(z)$ belongs to the big cell $\mathcal{C}$. Using Lemma 3.4 in the same way as in [DPW], one can show that this is true for all $z$, excepted maybe on a finite subset $\{a_1, \ldots, a_p\} \subset L$. The second property is proved also in [DPW]. The last one follows easily by writing

$$\mu_\lambda = F_\lambda^+ \left[ \alpha_\lambda - (F_\lambda^+)^{-1}dF_\lambda^+ \right] (F_\lambda^+)^{-1}$$

which implies on the one hand that $\mu_\lambda$ is in $\mathcal{H}_{-2,\infty}(L \setminus \{a_1, \ldots, a_p\})$, once one keep in mind the fact that $F_\lambda^+(z) \in \Lambda^+ G_\tau^\mathbb{C}$. But on the other hand $F_\lambda^-(z) \in \Lambda_\star^- G_\tau^\mathbb{C}$ and thus there is no nonnegative power of $\lambda$ in the Fourier expansion of $\mu_\lambda$. This implies the conclusion. ∎

# 4 A list of cases

## 4.1 The Euclidean space

The case of $\mathbb{C}^2$ has been thoroughly studied in a first article [HR1] including an explicit description of all tori. We will only point out the – obvious – differences between $\mathbb{C}^2$ and the other Hermitian symmetric spaces in the light of our study. Since the group of isometries is the semi-direct product $G = U(2) \ltimes \mathbb{C}^2$, we have the additional commutation property $[\mathfrak{m}, \mathfrak{m}] = 0$. The equations in (3) then decouple to yield a PDE on $H$ (in $\alpha_2, \alpha_0$) and a PDE on $\mathfrak{m}$ with parameters $\alpha_2, \alpha_0$. Moreover the only nonlinearity has disappeared ! Finally, one may go even further than the standard analysis of that case, since $d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] = 0$ implies the (local) existence of lifts gauging $\alpha_0$ to zero (we call them spinor lifts). At that point the problem of finding surfaces is equivalent to solving two linear PDEs (plus the Poincaré integration procedure to get $f$). But we do not need to use the coclosedness condition, and the commutation property is also the key point for the linear Weierstrass representation of Lagrangian surfaces in $\mathbb{C}^2$ – not only stationary ones. We derive a Dirac-type equation characterizing all such surfaces (see [HR2]).

## 4.2 The complex projective plane

We write the projective plane as a symmetric space

$$\mathbb{CP}^2 = G/H = \frac{SU(3)}{S(U(2) \times U(1))}$$

where[6]

$$S(U(2) \times U(1)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}, \ A \in U(2) \right\}$$

with Lie algebra

$$\mathfrak{h} = \mathfrak{sl}(3, \mathbb{C}) \cap (\mathfrak{u}(2) \oplus \mathfrak{u}(1)) = \left\{ \begin{pmatrix} X & 0 \\ 0 & -\operatorname{tr} X \end{pmatrix}, \ X \in \mathfrak{u}(2) \right\}.$$

Here and in subsequent sections, $\tilde{X}$ will denote the conjugate of $X$ with respect to the real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$; in the $\mathfrak{su}(3)$ case, $\tilde{X} = -X^*$.

The quotient map is given simply by $SU(3) \to \mathbb{CP}^2$, $g \mapsto \mathbb{C}g\epsilon_3$ where $\epsilon_3 = (0, 0, 1)$. The natural involution $\sigma$ acts on $SU(3)$ (and its differential on $\mathfrak{g} = \mathfrak{su}(3)$) by conjugation:

$$\sigma : \begin{pmatrix} A & u \\ -u^* & a \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & u \\ -u^* & a \end{pmatrix} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} A & -u \\ u^* & a \end{pmatrix}$$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ splits as the direct sum of the $+1$ eigenspace of $\sigma^2$, $\mathfrak{h}$, and the $(-1)$-eigenspace

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \ u \in \mathbb{C}^2 \right\}$$

identified with $\mathbb{C}^2$ via

$$\begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} \mapsto u.$$

The adjoint representation of $H$ on $\mathfrak{m}$ is surjective and almost effective:

$$\operatorname{Ad} \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix} = \left[ u \mapsto (\det A) A u \right]$$

and

$$\operatorname{ad} \begin{pmatrix} X & 0 \\ 0 & -\operatorname{tr} X \end{pmatrix} = \left[ u \mapsto (X + \operatorname{tr} X \mathbb{1}) u \right]$$

so that the complex structure is $\exp(\frac{\pi}{2} \operatorname{ad} Y)$ with

$$Y = \frac{i}{3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}.$$

---

[6]Notice that the action of $SU(3)$ is only almost effective, with a kernel made of three elements: the cubic roots of identity.

The order four automorphism acting on $G^{\mathbb{C}}$ is

$$\tau : g \mapsto \left( \begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1 \end{array} \right) {}^tg^{-1} \left( \begin{array}{cc|c} & -1 & \\ 1 & & \\ \hline & & 1 \end{array} \right). \tag{7}$$

Notice that on $G$ itself ${}^tg^{-1} = \bar{g}$; hence its differential acting on $\mathfrak{g}^{\mathbb{C}}$ (still denoted $\tau$) is:

$$\tau : X \mapsto - \left( \begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1 \end{array} \right) {}^tX \left( \begin{array}{cc|c} & -1 & \\ 1 & & \\ \hline & & 1 \end{array} \right).$$

We then have a direct sum $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_2^{\mathbb{C}} \oplus \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}}$ with $\mathfrak{g}_2^{\mathbb{C}} = \mathbb{C}Y$,

$$\mathfrak{g}_0 = \left\{ \left( \begin{array}{cc} X & 0 \\ 0 & 0 \end{array} \right); \ X \in \mathfrak{su}(2) \right\}$$

$$\mathfrak{g}_{-1}^{\mathbb{C}} = \left\{ \left( \begin{array}{cc|c} & & -ia \\ & & b \\ \hline -ib & a & \end{array} \right), \ a, b \in \mathbb{C} \right\}$$

and

$$\mathfrak{g}_1^{\mathbb{C}} = \widetilde{\mathfrak{g}_{-1}^{\mathbb{C}}} = \left\{ \left( \begin{array}{cc|c} & & ia \\ & & b \\ \hline ib & a & \end{array} \right), \ a, b \in \mathbb{C} \right\}.$$

**Example 4.1** THE REAL PROJECTIVE PLANE $\mathbb{RP}^2$ is immersed minimally in $\mathbb{CP}^2$ (and its double cover is the only minimal Lagrangian sphere, up to unitary isometries, see [Y]). Choose the stereographic projection from the southern pole as conformal coordinate chart. The fundamental lift (real-valued of course) is:

$$F(z) = \frac{1}{1 + |z|^2} \left( \begin{array}{ccc} 1 - x^2 + y^2 & -2xy & 2x \\ -2xy & 1 + x^2 - y^2 & 2y \\ -2x & -2y & 1 - x^2 - y^2 \end{array} \right)$$

The Maurer-Cartan form satisfies $\alpha_2 = 0$ and

$$(1 + |z|^2)\alpha \ = \ \underbrace{\left( \begin{array}{cc|c} & -i & \\ i & & \\ \hline & & \end{array} \right)}_{\alpha_0} (\bar{z}dz - zd\bar{z})$$

$$+ \underbrace{\left( \begin{array}{cc|c} & & 1 \\ & & -i \\ \hline -1 & i & \end{array} \right)}_{\alpha'_{-1}} dz + \underbrace{\left( \begin{array}{cc|c} & & 1 \\ & & i \\ \hline -1 & -i & \end{array} \right)}_{\alpha''_1} d\bar{z}.$$

The associated family is only a change of variable by rotation in the $z$-plane.

**Example 4.2** THE CLIFFORD TORUS is the quotient of the standard torus $S^1 \times S^1 \times S^1 \subset \mathbb{C}^3$ by the Hopf action; it can be conformally parametrized as $\mathbb{C}f$ where

$$
f(x+iy) = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{2ix} \\ e^{i(y\sqrt{3}-x)} \\ e^{-i(x+y\sqrt{3})} \end{pmatrix}
$$

and the fundamental lift is

$$
F(z) = \frac{1}{\sqrt{6}} \begin{pmatrix} 2ie^{2ix} & 0 & \sqrt{2}e^{2ix} \\ -ie^{i(y\sqrt{3}-x)} & i\sqrt{3}e^{i(y\sqrt{3}-x)} & \sqrt{2}e^{i(y\sqrt{3}-x)} \\ -ie^{-i(x+y\sqrt{3})} & -i\sqrt{3}e^{-i(x+y\sqrt{3})} & \sqrt{2}e^{-i(x+y\sqrt{3})} \end{pmatrix}
$$

with Maurer-Cartan form

$$
\alpha = \underbrace{\left( \begin{array}{cc|c} i & -1 & \\ -1 & -i & \\ \hline & & \end{array} \right) \frac{dz}{2} + \left( \begin{array}{cc|c} i & 1 & \\ 1 & -i & \\ \hline & & \end{array} \right) \frac{d\bar{z}}{2}}_{\alpha_0}
$$

$$
+ \underbrace{\left( \begin{array}{cc|c} & & 1 \\ & & -i \\ \hline -1 & i & \end{array} \right) \frac{dz}{\sqrt{2}}}_{\alpha'_{-1}} + \underbrace{\left( \begin{array}{cc|c} & & 1 \\ & & i \\ \hline -1 & -i & \end{array} \right) \frac{d\bar{z}}{\sqrt{2}}}_{\alpha''_1}.
$$

Note that $\alpha_2 = 0$, which agrees with the fact that the Clifford torus is minimal.

**Example 4.3** VACUUM SOLUTIONS are obtained by taking extended lifts $F_\lambda = \exp(zM_\lambda + \bar{z}\widetilde{M_\lambda})$ where $M_\lambda$ is a constant in $\Lambda G_\tau$. Equation (3) amounts to $[M_\lambda, \widetilde{M_\lambda}] = 0$ For further simplification ($M_\lambda$ being constant) we gauge the $\mathfrak{g}^{\mathbb{C}}_{-1}$ part to

$$
\frac{e^\rho}{2} \left( \begin{array}{cc|c} & & 1 \\ & & -i \\ \hline -1 & i & \end{array} \right).
$$

That yields a family parametrized by complex numbers $b, c$ such that $e^{2\rho} = 8\,\mathrm{Im}(\bar{b}c) > 0$

$$
M_\lambda = \begin{pmatrix} b - i\lambda^{-2}a & c & \lambda^{-1}\frac{e^\rho}{2} \\ c & -b - i\lambda^{-2}a & -i\lambda^{-1}\frac{e^\rho}{2} \\ -\lambda^{-1}\frac{e^\rho}{2} & i\lambda^{-1}\frac{e^\rho}{2} & 2\lambda^{-2}ia \end{pmatrix}, \quad a = -\frac{\bar{c}+i\bar{b}}{3}
$$

Minimal conformal immersions correspond to $a = 0$ so $c = ib$ and $e^\rho = 2\sqrt{2}|b|$

$$
M_\lambda = \begin{pmatrix} b & ib & \lambda^{-1}\sqrt{2}|b| \\ ib & -b & -i\lambda^{-1}\sqrt{2}|b| \\ -\lambda^{-1}\sqrt{2}|b| & i\lambda^{-1}\sqrt{2}|b| & 0 \end{pmatrix}
$$

and for $b = \frac{i}{2}$ we recognize the Clifford torus above. Variations in $|b|$ amount to trivial scale changes, but changes in the argument of $b$ yield different examples (not gauge-equivalent); these however may not be periodic.

### 4.3   Lagrangian cones in $\mathbb{C}^3$

We explain here how a similar formalism applies as well to Lagrangian cones in $\mathbb{C}^3$, and how this relates formally to the previous section (knowing that such cones are intimately associated to Lagrangian surfaces in $\mathbb{CP}^2$ as explained in §1.3). This association is not one to one, but to each Lagrangian surface corresponds exactly a circle of cones, namely the orbit under the Hopf action of any member. To make this relation visually explicit, we will overline with a ˘ the corresponding quantity in $S^5$; for instance, the Legendrian map $\check{f} : L \to S^5$ projects down to a Lagrangian map $f : L \to \mathbb{CP}^2$.

We view now $S^5$ as the reductive[7] space $\check{G}/\check{H} = U(3)/\check{H}$ where

$$\check{H} = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & 1 \end{array} \right), \ A \in U(2) \right\}$$

is the isotropy group of $\epsilon_3$. The quotient map is $g \mapsto g\epsilon_3$. We have the reductive splitting $\check{\mathfrak{g}} = \mathfrak{u}(3) = \check{\mathfrak{h}} \oplus \check{\mathfrak{m}}$ with

$$\check{\mathfrak{h}} = \left\{ \left( \begin{array}{cc} X & 0 \\ 0 & 0 \end{array} \right), \ X \in \mathfrak{u}(2) \right\}, \quad \check{\mathfrak{m}} = \left\{ \left( \begin{array}{cc} 0 & u \\ -u^* & ia \end{array} \right), \ a \in \mathbb{R}, \ u \in \mathbb{C}^2 \right\}.$$

The same order four automorphism $\tau$ (formally) as in the previous section (see formula (7)) acts on $U(3)$ and splits the Lie algebra $\check{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{gl}(3,\mathbb{C})$ into four eigenspaces: $\check{\mathfrak{g}}_0^{\mathbb{C}} \oplus \check{\mathfrak{g}}_2^{\mathbb{C}} \oplus \check{\mathfrak{g}}_{-1}^{\mathbb{C}} \oplus \check{\mathfrak{g}}_1^{\mathbb{C}}$:

$$\check{\mathfrak{g}}_2 = \mathbb{R}\check{Y} \oplus \mathbb{R}\check{Z} \text{ with } \check{Y} = \left( \begin{array}{ccc} i & & \\ & i & \\ & & 0 \end{array} \right), \ \check{Z} = \left( \begin{array}{ccc} 0 & & \\ & 0 & \\ & & i \end{array} \right)$$

and $\check{\mathfrak{g}}_0 = \mathfrak{g}_0$, $\check{\mathfrak{g}}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^{\mathbb{C}}$, $\check{\mathfrak{g}}_1^{\mathbb{C}} = \mathfrak{g}_1^{\mathbb{C}}$. Comparing with §4.2, the only differences are (i) $\check{\mathfrak{g}}_2$ is two dimensional and (ii) the complex structure changes from $Y$ to $\check{Y}$. Nota bene: the contact distribution is generated as the orbit under $U(3)$ of the subspace

$$\check{\mathfrak{p}} = \left\{ \left( \begin{array}{cc} 0 & u \\ -u^* & 0 \end{array} \right), \ u \in \mathbb{C}^2 \right\} \simeq \mathbb{C}^2,$$

endowed with the complex structure $\mathrm{ad}\,\check{Y}$. Obviously $\check{\mathfrak{p}}^{\mathbb{C}} = \check{\mathfrak{g}}_{-1}^{\mathbb{C}} \oplus \check{\mathfrak{g}}_1^{\mathbb{C}}$ and $\check{\mathfrak{m}} = \check{\mathfrak{p}} \oplus \mathbb{R}\check{Z}$.

Consider a Lagrangian cone $\mathcal{C}$; its link is Legendrian, namely satisfies that: (i) its tangent bundle lies in the contact distribution $\Pi$ ($T_x M \perp ix$) and (ii) the tangent space $T_x M$ is Lagrangian in $\Pi_x$. Letting $\check{f} : L \to S^5$ be a conformal parametrization of $M$, conditions (i) and (ii) above amount to the existence of a (unique) fundamental lift $\check{F} \in U(3)$ such that:

$$d\check{f}_z = e^{\rho(z)}\check{F}(z)(\epsilon_1 dx + \epsilon_2 dy) \tag{8}$$

which can be rewritten in terms of $\check{\alpha} = \check{F}^{-1}d\check{F}$

$$\check{\alpha}\epsilon_3 = e^\rho(\epsilon_1 dx + \epsilon_2 dy) = e^\rho(\epsilon dz + \tilde{\epsilon}d\bar{z}) \tag{9}$$

---

[7]but not symmetric.

where

$$\epsilon_1 = \left( \begin{array}{cc|c} & & 1 \\ & & 0 \\ \hline -1 & 0 & \end{array} \right), \ \epsilon_2 = \left( \begin{array}{cc|c} & & 0 \\ & & 1 \\ \hline 0 & -1 & \end{array} \right),$$

$$\epsilon = \frac{\epsilon_1 - i\epsilon_2}{2} = \frac{1}{2} \left( \begin{array}{cc|c} & & 1 \\ & & -i \\ \hline -1 & i & \end{array} \right), \ \tilde{\epsilon} = \frac{1}{2} \left( \begin{array}{cc|c} & & 1 \\ & & i \\ \hline -1 & -i & \end{array} \right).$$

Recall that $\det_{\mathbb{C}} \check{F} = e^{i\beta}$ where $\beta$ is the Lagrangian angle. Extending to the orbit under the gauge action of $C^\infty(L, \check{G}_0)$ with $\check{G}_0 = \check{H} \cap SU(3) \simeq SU(2)$, we define Lagrangian lifts by the property that $\check{\alpha}_{\mathfrak{m}} = \check{\alpha}'_{-1} + \check{\alpha}''_1$. Notice that the condition is more complicated here because we need to assume that the map is horizontal (i.e. lies in the contact distribution), which excludes components along $\check{Z}$.

We can now characterize Hamiltonian stationary Lagrangian cones either intrinsically through the following

**Theorem 4.4** *A conical Lagrangian singularity (whose intersection with $S^5$ is conformally parametrized) is exactly obtained by integrating a flat $\mathfrak{u}(3)$-valued 1-form $\omega$ satisfying $\check{\alpha}_{\mathfrak{m}} = \check{\alpha}'_{-1} + \check{\alpha}''_1$ (hence $\check{\alpha}_2$ lies in $\mathbb{C}\check{Y}$ since $\mathbb{R}\check{Z} = \check{\mathfrak{m}} \cap \check{\mathfrak{g}}_2$). Furthermore, the immersion is H-minimal (resp. special Lagrangian) if $\alpha_2$ is coclosed (resp. vanishes).*

Denoting $\Lambda\mathfrak{u}(3)_{\tau,\mathfrak{p}}$ the subspace of the twisted loop-algebra where $Z$-part vanishes, there is an interesting bijective correspondence between this subspace and the loop algebra $\Lambda\mathfrak{su}(3)_\tau$, mapping flat extended connection forms to flat extended connections forms, which leaves all matrix coefficients unchanged but for the $\check{\mathfrak{g}}_2$ part where the complex structure $\check{Y}$ is mapped to complex structure $Y$:

$$\left( \begin{array}{ccc} a & & \\ & b & \\ & & 0 \end{array} \right) \mapsto \frac{1}{3} \left( \begin{array}{ccc} 2a-b & & \\ & 2b-a & \\ & & 0 \end{array} \right).$$

Or one can associate to the cone the projected Lagrangian surface in $\mathbb{CP}^2$ with the following data: a map $f = \mathbb{C}\check{f} : L \to \mathbb{CP}^2$ with a Lagrangian lift $F$. We claim that $F = e^{i\beta/3}\check{F}$ is such a lift. Indeed $F$ lifts $f$ since $\mathbb{C}F\epsilon_3 = f$ and $\det F = 1$. To prove that $F$ is Lagrangian consider its Maurer-Cartan form

$$\alpha = \check{\alpha} + \frac{d\beta}{3}\mathbb{1} = \check{\alpha}_0 + \check{\alpha}'_{-1} + \check{\alpha}''_1 + \underbrace{\frac{d\beta}{2}\check{Y} - \frac{id\beta}{3}\mathbb{1}}_{\frac{d\beta}{2}Y}.$$

Obviously $\alpha''_{-1} = 0$. Furthermore we see that the Lagrangian angle of the cone is equal to the Lagrangian angle of the surface in $\mathbb{CP}^2$. It may be noted that the fundamental lift is mapped thus to the fundamental lift.

## 4.4 The complex hyperbolic plane

The non compact dual of $\mathbb{CP}^2$ is the complex hyperbolic space

$$\mathbb{CD}^2 = SU(2,1)/S(U(2) \times U(1))$$

where

$$SU(2,1) = \left\{ g \in SL(3,\mathbb{C}), \ gBg^* = B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}$$

with Lie algebra

$$\mathfrak{su}(2,1) = \left\{ \begin{pmatrix} X & v \\ v^* & -\operatorname{tr} X \end{pmatrix}, \ X \in \mathfrak{u}(2), \ v \in \mathbb{C}^2 \right\}.$$

The same automorphism $\tau$ acts on $M$.

## 4.5 $\mathbb{CP}^1 \times \mathbb{CP}^1$

We consider the following Hermitian symmetric space:

$$\mathbb{CP}^1 \times \mathbb{CP}^1 = \frac{SU(2) \times SU(2)}{U(1) \times U(1)}$$

($G = SU(2) \times SU(2)$ will be written as bloc-diagonal four by four matrices) and define an order four automorphism $\tau : g \mapsto TgT^{-1}$ where

$$T = \left( \begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array} \right).$$

Its differential at identity diagonalizes on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ with eigenspaces $\mathfrak{g}_0^{\mathbb{C}} = \mathbb{C}X$, $\mathfrak{g}_2^{\mathbb{C}} = \mathbb{C}Y$ (as usual $Y$ is the complex structure)

$$X = \frac{1}{2} \left( \begin{array}{cc|cc} -i & & & \\ & i & & \\ \hline & & i & \\ & & & -i \end{array} \right), \qquad Y = \frac{1}{2} \left( \begin{array}{cc|cc} -i & & & \\ & i & & \\ \hline & & -i & \\ & & & i \end{array} \right),$$

$$\mathfrak{g}_{-1}^{\mathbb{C}} = \left\{ \left( \begin{array}{cc|cc} & v & & \\ u & & & \\ \hline & & & iu \\ & & iv & \end{array} \right) u,v \in \mathbb{C} \right\},$$

$$\mathfrak{g}_1^{\mathbb{C}} = \left\{ \left( \begin{array}{cc|cc} & v & & \\ u & & & \\ \hline & & & -iu \\ & & -iv & \end{array} \right) u,v \in \mathbb{C} \right\}.$$

## 4.6 The non compact dual of $\mathbb{CP}^1 \times \mathbb{CP}^1$

As expected, the situation is very close to its compact dual.

$$M = G/H = \frac{SU(1,1) \times SU(1,1)}{U(1) \times U(1)}$$

the automorphism has the same expression and so do the eigenspaces.

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